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CONSTRUCTING PERIODIC CURVES WITH AN EXPONENTIAL-ALGEBRAIC HYBRID INTERPOLATING POLYNOMIAL

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By "periodic" we mean a parametric curve that replicates itself on some predefined period up to a translation. A good example of a parametric curve is a helix. The domain of the function that sets the parametric curve is real numbers, the codomain is then points in n-dimensional space. The period can be an arbitrary number, but from now on in this work we'll always presume that the period is 2π.
In this work, we propose a way to construct a periodic curve that goes through a set of predefined points and, while doing so, provides additional control over its shape. To build such a periodic curve, we use an exponential-algebraic

hybrid polynomial that is partially similar to the Fourier polynomial in complex exponential notation. The difference is that the hybrid polynomial we propose also has algebraic terms, moreover while exponential terms inherently require imaginary numbers as their input, the algebraic terms do not. This polynomial interpolates a set of complex numbers, each assigned to a single real parameter. The real parts of the complex data points constitute the points in n-dimensional space the target curve passes through, while the imaginary parts of the same data points affect the shape of the curve without compromising the interpolating properties of the polynomial in real numbers.
In this way, we effectively split our input in two. The "curve goes through the points" conditions are set by the real parts of the complex input data points, and we can then use the imaginary parts to control the curve's shape. An example of such control is minimizing the curve's length on a given parameter range.

ПОБУДОВА ПЕРІОДИЧНИХ КРИВИХ КОМБІНОВАНИМ ЕКСПОНЕНЦІЙНО-АЛГЕБРАЇЧНИМ ІНТЕРПОЛЮЮЧИМ ПОЛІНОМОМ

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Ключові слова: *інтерполяція, дотичні вектори, поліном Фур'є, чисельна оптимізація, неперервна оптимізація, періодична крива, замкнена крива.*

Під «періодичною» кривою матимемо на увазі криву, що повторює себе на певному періоді з точністю до переносу. Прикладом такої криви є гвинтова лінія. Доменом функції, яка задає таку параметричну криву, є множина дійсних чисел, а кодоменом – множина всіх точок n-вимірного простору. Періодом такої кривої може бути будь-яке довільне дійсне число, але тут і надалі вважатимемо періодом періодичної кривої 2π.
У цій роботі пропонується спосіб побудови періодичної кривої, що проходить через наперед задану множину точок і водночас забезпечує додатковий контроль за формою. Для побудови такої кривої використовується комбінований експоненційно-алгебричний поліном, частково подібний до поліному Фур'є в експоненційному записі. Різниця між запропонованим поліномом і поліномом Фур'є полягає у тому, що комбінований поліном містить разом з експоненційними і алгебричні члени, причому, якщо експоненційні члени потребують уявного аргументу, алгебричні приймають аргумент у дійсному просторі. Такий поліном інтерполює множину комплексних чисел, призначених до унікальних значень дійсного параметра. Дійсні частки таких чисел відповідають точкам кривої у дійсному просторі, через які крива проходитиме, а уявні – впливають на форму кривої не віднімаючи умови проходження через задані точки завдяки інтерполяційним властивостям полінома.
У такий спосіб вхідні дані фактично розділяються на дві множини. Умова проходження кривої через задані точки задається дійсними частками вхідних даних, а уявні частки тих чисел додатково впливають на форму кривої. У просторі уявних часток вхідних даних можлива чисельна оптимізація певних властивостей кривої, наприклад мінімізації довжини кривої на параметричному інтервалі її періоду.

Literature review. For the most practical applications, the problem of periodic curves construction narrows down to several distinct types of curves: helical polynomial space curves as a special case of Pythagorean-hodograph curves [1], slant helix curves [2; 3] or conical geodesic curves [4]. This specialization is attributed to the properties of helical curves we want to sustain for instance the constant precession [5].

For the more generic periodic curves, it is often practical to use splines which can also preserve some desirable geometric invariants. For instance, rational splines can represent circles exactly [6; 7] and a circle is a special case of a periodic curve. Also, there are splines based on Pythagorean-hodograph curves [8] that can be used for periodic curve construction as well.

For the most generic periodic curves, when no specific invariants are expected to hold and no regularity of smoothness only the C^n continuity is necessary, we can also use non-uniform rational basis splines [9], or any rational or even polynomial splines at all, and emulate the periodicity by replicating the same curve segment with the same translation for as long as needed.

Introduction. In our earlier paper [10] we proposed the following polynomial for constructing closed curves with given properties:

$$P(x) = \sum_{k=0}^{n} a_k e^{ikx}.$$

When x is real, the exponential argument in every polynomial term is the real x multiplied by a real number k and imaginary entity i, therefore, becoming fully imaginary. Then by Euler's formula, the polynomial decomposes into a set of sinusoids and cosinusoids, therefore becoming a periodic function. The coefficients a_k, however, are complex.

This polynomial can be interpolating for a set of exactly $(n + 1)$ distinctive data points (t, p) where t is a real number, and p is complex. This polynomial is equivalent to the Fourier polynomial only of $2n + 1$ members where the constant term is also a complex number. Normally, this Fourier polynomial would have been excessive for $n + 1$ points interpolation, but by splitting the input into real and imaginary parts, we only use the real part as interpolating data, and we retain the control over the imaginary part to have additional controls over the interpolating function.

The interpolating polynomial is then used to build the curve that both goes through a set of points and allows additional control over its properties. This additional control is valuable in practice. The main goal of the current work is to extend the proposed solution from closed curves to open periodic ones.

The auxiliary goal of this work is to explore whether optimizing the curve shape by minimizing its length in complex space can help approach the construction of minimal periodic surfaces. We use numeric methods, so we wouldn't expect true minimal surfaces to emerge but only some reasonable approximations.

Methods. We use the analytic method to build interpolating polynomials along with their derivatives

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in real numbers but continuous numeric optimization in imaginary parts of the input data points to optimize non-interpolating properties. Translated into geometric terms, we grant that the curve passes exactly through the selected points and has exact tangent vectors in them, but other properties, such as curve length, are only optimized as far as numeric methods let us. Consequently, this means that the curve length is not granted to be minimal but only not larger than it would appear without numeric optimization.

We collect the evidence for the proposed solution with computational experiments and use statistical analysis to process the results. For the specific methodology regarding the minimal surface construction, please refer to the corresponding section of the paper.

The exponential-algebraic hybrid interpolating polynomial

Using the periodic exponential polynomial, we can construct closed curves with predefined properties. Closed curves are periodic too, but they replicate themselves on a certain period without any translation. To introduce this translation, we need to add non-periodic algebraic terms to our interpolating polynomial. In this paper, we propose to add two algebraic members: a_1x and a_0. The exponential-algebraic hybrid polynomial will then look like this:

$$P(x) = \sum_{k=2}^{n} a_k e^{ikx} + a_1 x + a_0.$$

Some other problems might require adding other members of different powers, but for adding the translation to the curve, linear members are enough. Also, the a_0 member is just a constant and it's equivalent to $a_k e^{ixk}$ when $k = 0$, so it's also present in the original polynomial so calling it algebraic and not a Fourier member is ambiguous.

To construct a periodic curve with this polynomial, we define m points in space: (x_j, y_j, z_j), $j = 1..m$ and associate their respective parameter values on a target curve t_j, $j = 1..m$. Then by interpolating this data set one coordinate at a time, we get a triplet of polynomials that set the points of a curve $P(t) = (P_x(t), P_y(t), P_z(t))$. The interpolation conditions are as follows:

$$P_x(t_j) = x_j,\ j = 1..m,$$
$$P_y(t_j) = y_j,\ j = 1..m,$$
$$P_z(t_j) = z_j,\ j = 1..m.$$

Each of the coordinates gives us a system of linear equations. If $m = n + 1$, then each of the systems, given that none of the points are assigned to the same parameter value, has a unique solution. These solutions constitute the coefficients for the hybrid polynomials: $P_x(t)$, $P_y(t)$, and $P_z(t)$.

The curve $P(t) = (P_x(t), P_y(t), P_z(t))$ is periodic, but in general case isn't closed. To be closed, it would need to have the a_1 coefficient for every per-coordinate polynomial equal to zero. An example of an open periodic curve (constructed in 2D space for ease of visualization) is shown in Figure 1.

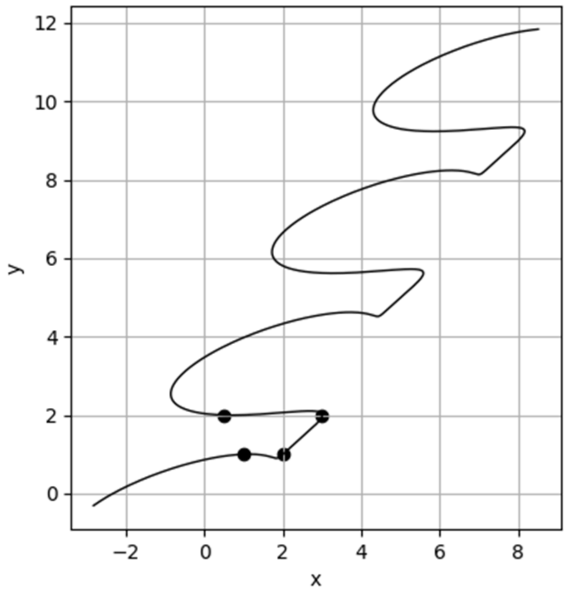

Fig.1. An example of an open periodic curve that passes through 4 points built with a hybrid polynomial

Noteworthy, a curve represented by the hybrid polynomial is the closest to a helix when it passes through 3 and not 4 points (Figure 2), although the helix itself is typically set by four points in space. This may be attributed to the redundancy of the hybrid polynomial, the same redundancy we will exploit later.

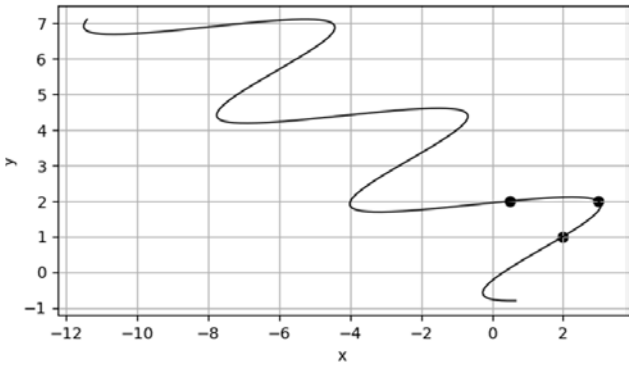

Fig.2. An example of a periodic curve constructed with a hybrid polynomial that passes through 3 points

In our earlier work, we added the derivatives conditions into the linear systems to ensure that the target curve not only passes through some points but also has tangent vectors defined in some points [10]. This provides even greater control over the shape of the curve. The level of control is similar to what Bezier curves or Hermit splines provide.

To achieve this control, we need to add the polynomial's derivatives equating the tangent vectors' coordinates to the system of equations:

$$P_x(t_p)` = dx_p,$$

$P_y(t_p)` = dy_p,$
$P_z(t_p)` = dz_p.$

The derivative $P(x)`$ of the hybrid polynomial $P(x)$ is:

$$P(x) = \sum_{k=2}^{n} a_k i(k-1)e^{ikx} + a_1.$$

The tuple (dx_p, dy_p, dz_p) – is the tuple of coordinates of the tangent vectors at the points of the curve that correspond to the parameter values t_p, where $p = 1..q$, for a set of q tangent vectors.

To keep the systems solvable we must compensate for the addition of new conditions by either increasing the polynomials' degree or reducing the number of interpolation points. For instance, in Figure 3 we show an example of a periodic curve that goes through 2 points and has 2 tangent vectors defined in these points. The degree of this polynomial is then 3.

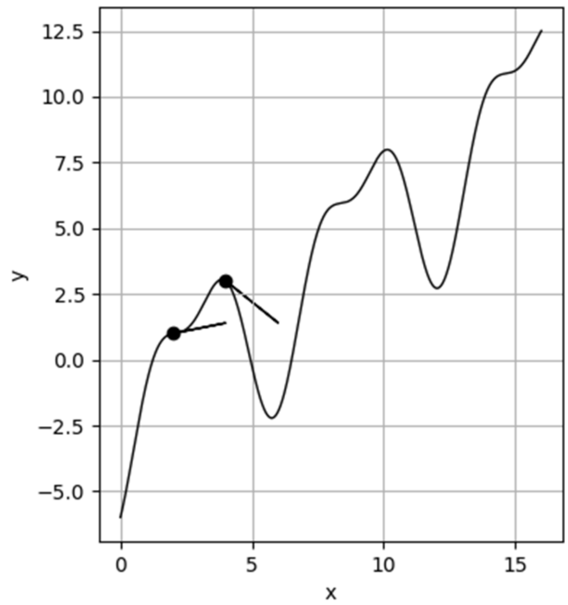

Fig. 3. An example of a periodic curve that passes through a pair of points and has also tangent vectors defined at those points

For now, we have only set the real parts of the interpolation data points, using the points' or tangent vectors' coordinates as real numbers to set the systems of linear equations when computing the interpolating polynomial coefficients. But we can add imaginary parts to all the numbers that are based on the real coordinates as well. This approach adds variability to the curve's shape without compromising the point or tangent conditions.

Let us restate the systems of equations to include the imaginary components xim_j, yim_j, and zim_j which we add to the coordinates of the interpolating points (x_j, y_j, z_j). From now on, we'll consider these as the systems of equations that connect the coefficients of the polynomials with the select conditions:

$Pc_x(t_j) = x_j + xim_j, j = 1..m,$
$Pc_y(t_j) = y_j + yim_j, j = 1..m,$
$Pc_z(t_j) = z_j + zim_j, j = 1..m.$

Solving these systems, when $m = n + 1$, results in the coefficients of the per-coordinate interpolating polynomials we can use to construct the periodic curve passing through m points for any values of the imaginary components xim_j, yim_j, and zim_j.

An example of curves that pass through the given set of points in real numbers but have different values of imaginary components is shown in Figure 4.

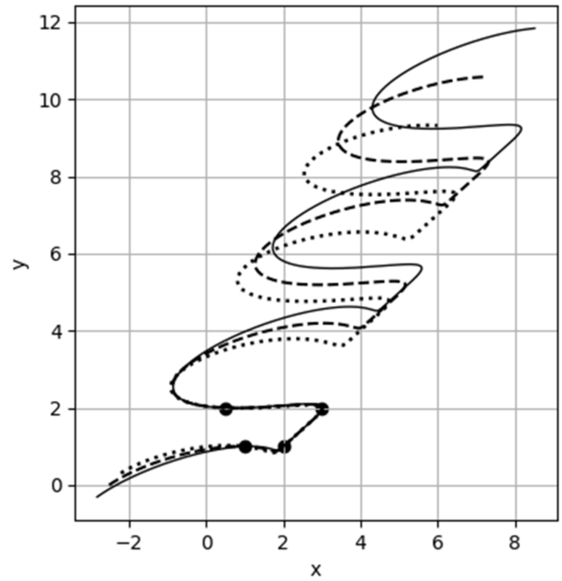

Fig. 4. An example of curves that go through the same 4 points but remain variable due to the imaginary components of the data points

Imaginary components of the points' coordinates that do not affect the interpolation conditions form a separate space in which the curve, while constrained to the interpolating points, is variable. We can exploit this to numerically optimize some properties of the curve by finding the optimal point in this separate space made of the imaginary components of the input data: $[xim_1, yim_1, zim_1, xim_2, yim_2, zim_2, \dots, xim_j, yim_j, zim_j, \dots, xim_m, yim_m, zim_m]$.

For instance, we can minimize the length of the curve of a parameter's interval [0.. 2π] using a numerical optimization method, such as the Broyden, Fletcher, Goldfarb, and Shanno [11] method. The length of the curve for the target function can then also be computed numerically. Since the curve is periodic, the minimization of its length on the period-long parameter interval means also that the length will be minimized on every interval longer than 2π. Further in this paper, when mentioning minimizing the curve's length we specifically mean the minimization of a periodic curve's length on the parameter's interval [0.. 2π].

An example of such a curve with minimized length is shown in Figure 5.

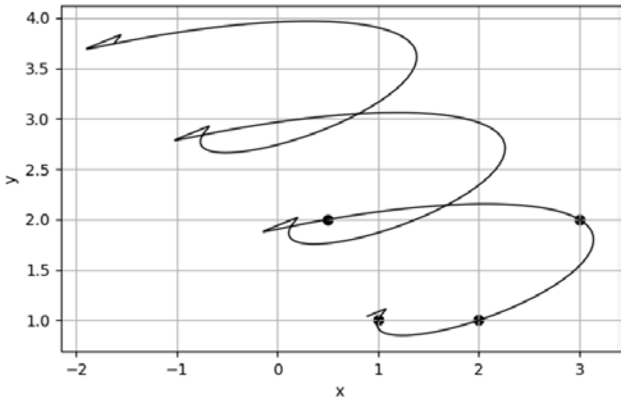

Fig. 5. An example of a curve with a minimized length that goes through 4 points

The points through which the curve passes in Figure 5 are the same points as in Figures 3 and 4. This is the same data set in real numbers, the same degrees of polynomials that define the curve, the only thing that differs is the imaginary parts of the data and the polynomial coefficients respectively.

Building a periodic surface based on a periodic curve

It is well known that if the length of a parametric curve $P(t)$ in the complex space equals 0, then the surface $S(u, v)$ formed from this curve by the parameter substitution will be minimal. The parameter substitution in question is this:

$$t = u+iv.$$

The methods to construct such surfaces are studied in different works of Ausheva N. M. such as [12; 13].

We use the same substitution to turn a periodic curve built with the hybrid exponential-algebraic polynomial into a periodic surface (see Figure 6). We will call the curve that forms such a surface a forming curve.

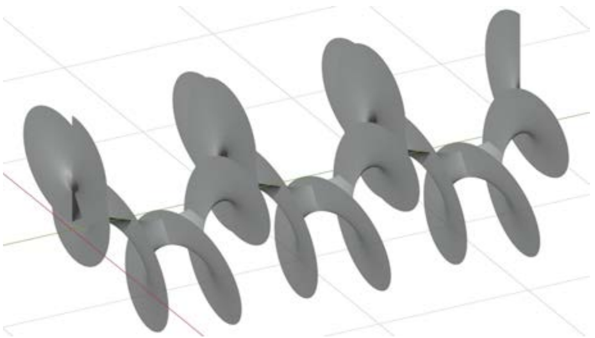

Fig. 6. A fragment of a periodic surface constructed with a periodic forming curve

A periodic surface is then the surface that replicates itself on some interval of parameter u (Sic! not v) up to some translation. A helical surface, or a surface of a screw, is an example of such a periodic surface.

The question arises whether minimizing the length of the forming parametric curve to any non-zero number will minimize the surface's curvature up to any kind of proportion. Given that we use computer-based numeric optimization to build the forming curve, we arranged a series of computational experiments to answer this question.

Forming curve length minimization in the complex space

Let $C(t) \to (x, y, z)$ be a parametric curve defined with a hybrid polynomials $C(t) = (P_x(t), P_y(t), P_z(t))$. The curve passes through a predefined set of points (x_j, y_j, z_j), $j = 1..m$ in the real 3-dimensional space $\mathbb{R}^3$. We use numeric optimization to minimize its length in the complex space $\mathbb{C}^3$ by finding the optimum point in the space of imaginary parts of the interpolating points:

$[xim_1, yim_1, zim_1, xim_2, yim_2, zim_2, \ldots, xim_j, yim_j, zim_j, \ldots, xim_m, yim_m, zim_m] \in \mathbb{R}^{3m}$.

When substituting t with $u + iv$, the curve equation becomes the surface equation:

$$S(u, v) \to (x, y, z).$$

We conducted a series of experiments where the curve goes through the set of m points produced by the randomization of the points equidistantly parametrized on a spiral curve on the parameter range $t = [0, 2\pi]$. The z coordinate of such points is randomly shifted within the predefined $[-dxyz, dxyz]$ range, their parameter is shifted randomly within the $[-dt, dt]$ range.

In each experiment, we measured the average of the mean curvature values (AMC) of the surface S defined randomly in a number of control points in the parametric space on the $u = [0, 2\pi]$, $v = [0, 1]$ range. The number of control points is variable. Let's call it n.

Minimization of the curve was conducted numerically using the Broyden, Fletcher, Goldfarb, and Shanno method. We tried to find the dependency of this measured AMC on the minimized length of the forming curve.

We conducted a set of experiments with $m = 5$ interpolating points, and $n = 1000$, 10000, and 100000 control points. The first two experiments were run for 100 randomized combinations of interpolating points, the last experiment had to be stopped at 50 randomized configurations due to its excessive runtime.

The data collected has been analyzed in the following way. To establish the correlation between the curve length minimization in the complex space and the minimization of AMC of the surface, the Pierson coefficient has been computed. Also, the p-value has been computed to validate the zero hypothesis.

The results were inconclusive. On one hand, some experiments show the expected correlation between the forming curve's length in the complex space and the

AMC computed, however, the correlation is too weak to state that minimizing the complex length of the forming curve leads to the minimization of the AMC.

The p-values we computed confirm this conclusion for each series of experiments.

We tried an alternative way of minimizing the AMC, which lies in numerically minimizing AMC on a finite set of points defined in a regular grid on $u = [0, 2\pi]$, $v = [0, 1]$. To minimize this metric we had to employ another optimization algorithm: Powell's conjugate direction [14].

But even with this problem restatement, we didn't get criteria for guaranteed AMC minimization. Moreover, even the correlation between the AMC of the points on a regular grid and the AMC of the randomized control point has not been confirmed statistically.

Conclusion

Constructing a periodic curve with the hybrid polynomial of degree n, consisting of two algebraic members: a_1x and a_0 and n-2 exponential members $a_k e^{ix(k-1)}$ allows passing the curve through the n points of the real 2D space (x_i, y_i) or 3D space (x_i, y_i, z_i), $i = 1..n$. It also enables the numerical optimization of a predefined metric in the space of the imaginary components xim_i, yim_i, zim_i of the interpolating points ($x_i + i\ xim_i$, $y_i + i\ xim_i$, $z_i + i\ zim_i$).

However, this is not sufficient to construct minimal surfaces or even surfaces with minimized mean curvature. While we can predictably minimize the length of a forming curve in the complex space, this minimization does not necessarily correlate with a predictable reduction in the mean curvature of the resulting surface on the intervals $u = [0, 2\pi]$, $v = [0, 1]$.

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